Year 12 Mathematics

EAS 2.7

Calculus Methods

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Contents

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Calculus Methods 2.7

This achievement standard involves applying calculus methods in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply calculus methods in solving problems.	• Apply calculus methods, using relational thinking, in solving problems.	• Apply calculus methods, using extended abstract thinking, in solving problems.

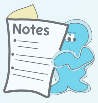
This achievement standard is derived from Level 7 of The New Zealand Curriculum and is related to the achievement objectives

- sketch the graphs of functions and their gradient functions and describe the relationship between these graphs
- apply differentiation and anti-differentiation techniques to polynomials.
- Apply calculus methods in solving problems involves:
  - selecting and using methods
  - demonstrating knowledge of calculus concepts and terms
  - communicating using appropriate representations.
- Relational thinking involves one or more of:
  - selecting and using a logical sequence of steps
  - connecting different concepts or representations
  - demonstrating understanding of concepts
  - forming and using a model;

  and also relating findings to a context, or communicating thinking using appropriate mathematical statements.
- Extended abstract thinking involves one or more of:
  - devising a strategy to investigate a situation
  - demonstrating understanding of abstract concepts
  - developing a chain of logical reasoning, or proof
  - forming a generalisation;

  and also using correct mathematical statements, or communicating mathematical insight.
- Problems are situations which provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.
- Methods include a selection from those related to:
  - derivatives and anti-derivatives of polynomials given in expanded form
  - gradient functions
  - gradient at a point
  - equation of a tangent
  - turning points where $f'(x) = 0$ and their nature
  - function from a derived function
  - rate of change problems (such as kinematics).

The Concept of Gradient



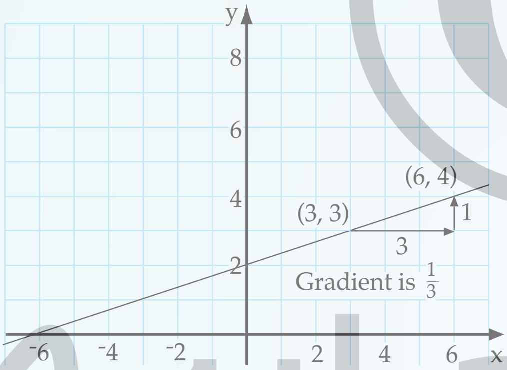
Gradient of a Straight Line

The steepness of a graph is called the gradient. With a straight line the entire line has the same gradient.

With a straight line we calculate the gradient by locating two coordinate pairs in the line and using

$$\text{Gradient} = \frac{\text{change in vertical}}{\text{change in horizontal}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$



For this straight line $(x_1, y_1) = (3, 3)$ and $(x_2, y_2) = (6, 4)$

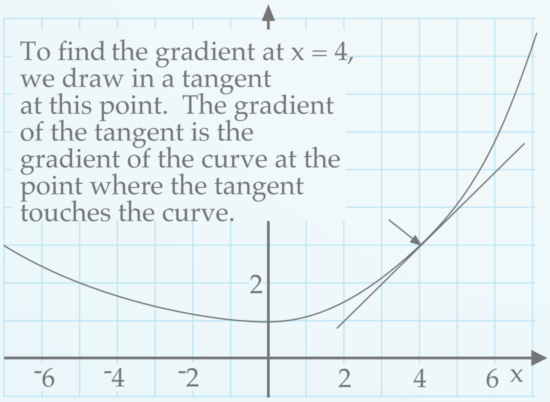
$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

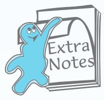
$$= \frac{4 - 3}{6 - 3}$$

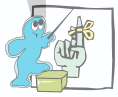
$$= \frac{1}{3}$$

Gradient of a Curved Line

With a curved line the steepness or gradient is changing constantly, so we find the gradient at a point by finding the gradient of a tangent at that point.







The concept of a tangent assumes the line is going from left to right.

If the line is going up the gradient is positive.

If the line is going down the gradient is negative.

If the line is flat the gradient is zero.

If the line is vertical the gradient is undefined.

Remember the definition of vertical change over the horizontal change is the same as the definition of the tangent of an angle.

$$\tan A = \frac{\text{vertical}}{\text{horizontal}}$$

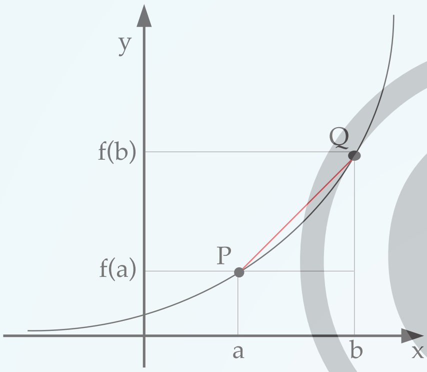
$$\text{Gradient} = \frac{\text{vertical}}{\text{horizontal}}$$

$$\tan A = \text{Gradient}$$

The Derivative from First Principles

The Derivative

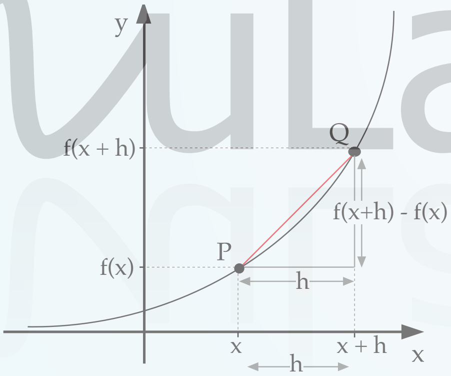
We have already established that the average rate of change of a graph is given by finding the gradient of a chord.



To calculate the gradient or average rate of the chord PQ we use the formula

$$m = \frac{f(b) - f(a)}{b - a}$$

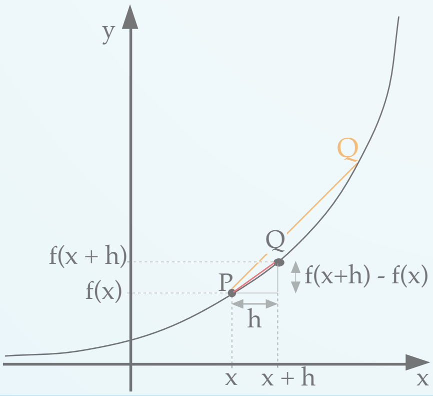
If we let the gap between a and b be h and we change our notation so P is now (x, f(x)) then Q becomes (x + h, f(x + h))



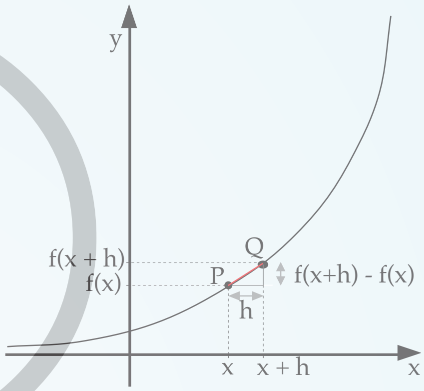
The gradient of the chord is now

$$m = \frac{f(x+h) - f(x)}{(x+h) - x}$$

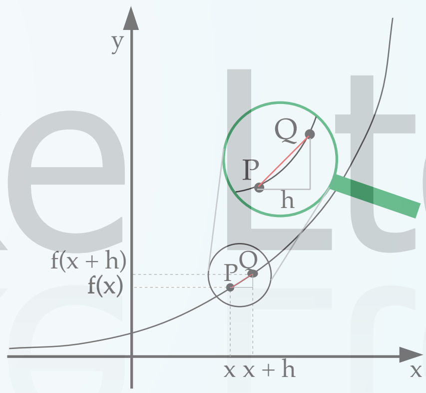
Now if we reduce the value of h, then Q gets closer to P.



If we continue to reduce the value of h then the gradient of the chord PQ gets closer and closer to the gradient of a tangent at the point P.



If the interval over which we are finding the average rate of change is made smaller and smaller, then the



limiting value of the chord's gradient as h approaches zero $[h \to 0]$ is the gradient of the tangent and the gradient of the curve at this point.

This limiting value of the gradient at a point becomes

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This derived function f′(x), is known as the **gradient function** and the process is called **differentiation**.

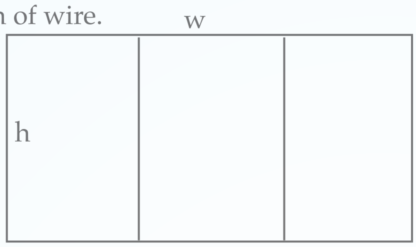
When we use the formula above to calculate the derived function we are **differentiating by first principles**.

The notation f′(x) is used to denote the gradient function.

Example

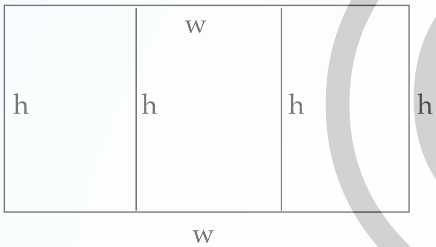
A wire frame is to be constructed out of 1200 mm of wire.

Find the height and width of the frame so that the total area is as large as possible.



Using **DELS DEBT**

Draw a diagram.



Express the problem that needs to be maximised or minimised (often in terms of two variables).

$$\text{Area} = \text{height} \times \text{whole width}$$
$$= h \times w$$
$$= hw$$

Link the two variables above with an equation.

$$1200 = 4h + 2w$$

Substitute for one variable using the link above.

$$h = \left(\frac{1200 - 2w}{4}\right)$$
$$= 300 - 0.5w$$

$$\text{Area} = w(300 - 0.5w)$$
$$= 300w - 0.5w^2$$

Differentiate.

$$\text{Area}' = 300 - w$$

Equate to zero so that you can find the turning points.

$$300 - w = 0 \qquad \text{For Max / Min}$$
$$w = 300$$

Back substitute into the original problem to find the maximum or minimum value.

$$\text{Area} = 300 \times 300 - 0.5(300)^2$$
$$= 45\,000 \text{ mm}^2$$

Test your answer by re-reading the question. No. The original question called for the height h and width w, so the correct answer is

$$w = 300 \text{ mm}$$

and

$$h = 300 - 0.5 \times 300$$
$$= 150 \text{ mm}$$

Extra Examples

Merit / Excellence – Find the maximum solution for each of the following using differentiation.

116. The sum of two numbers is 31.
Find the maximum product of these two numbers.

Let the numbers be x and y

Draw a diagram (not needed)

Express the problem

prod $= x.y$

Link the two variables

Sum $x + y = 31$

Substitute

Differentiate

Equate to 0 and solve

Back substitute

Test

117. A farmer has 1000 m of electric fence with which to make a rectangular paddock. What is the largest area he can enclose?

Draw a diagram

Express

Area =

Link

1000 =

Substitute

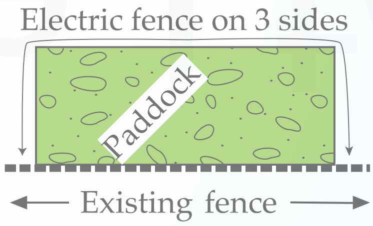
Differentiate

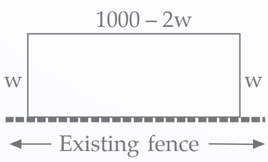
Equate

Back substitute

Test

118. The farmer in the previous question decides to use an existing fence as one of the boundaries of his paddock.
What is the largest rectangular paddock he can now enclose with 1000 m of electric fence on the remaining three sides?





The Reverse of Differentiation – Anti-differentiation

Anti-differentiation

Differentiation produces a rate of change or gradient function.

If we start with a derivative or rate of change function and reverse the process, we get the anti-derivative (or integral).

We examine a derivative to identify the process.

For $f(x) = ax^n$

Then $f'(x) = nax^{n-1}$

We multiply by the old power and reduce the power by 1.

In reverse this becomes

Increase the power by one and divide by the new power.

For the derived function

$$f'(x) = bx^m$$

The anti-derivative is

$$f(x) = \frac{b}{m+1}x^{m+1} + C$$

For example if

$$f'(x) = x^3$$

then $f(x) = \frac{1}{4}x^4 + C$

or if $g'(x) = 8x - 3$

then $g(x) = \frac{8}{1+1}x^{1+1} - \frac{3}{0+1}x^{0+1} + C$

$= 4x^2 - 3x + C$

We need to add a constant, C, every time we find an anti-derivative as when we differentiate a constant it goes to zero so when we are finding the anti-derivative we are no longer able to determine the value of any constant.

We sometimes use the $\int$ symbol to show we are finding the anti-derivative.

$$f(x) = \int f'(x)\,dx.$$

Indefinite Integral

The process of anti-differentiating is also called integration. The symbol $\int$ is used to denote the integral and dx (which comes from Leibniz notation) tells us which variable we are finding the anti-derivative of. For example if F(x) is defined as the integral of f(x), then

$$F(x) = \int f(x)\,dx$$

It is expected in NCEA 2 that references to this process will be referred to as anti-differentiation and the term integration will not be used.

Extra Notes

The constant of integration

Any constant when differentiated is zero, so when anti-differentiating it is not possible to identify any constants.
Demonstrating this, if

$$f(x) = 3x^2 + 4x + 19$$
$$f'(x) = 6x + 4$$

and if $f(x) = 3x^2 + 4x - 43$

$$f'(x) = 6x + 4$$

Therefore if we start with $f'(x) = 6x + 4$ we do not know the correct anti-derivative. It could be $f(x) = 3x^2 + 4x + 19$, $f(x) = 3x^2 + 4x - 43$ or $f(x) = 3x^2 + 4x +$ any constant.

Therefore every time an expression is anti-differentiated a constant , usually C, is added at the end.

$f'(x)$	$f(x)$
k	$kx + C$
bx	$\frac{b}{2}x^2 + C$
$ax^2 + bx$	$\frac{a}{3}x^3 + \frac{b}{2}x^2 + C$
x^m	$\frac{1}{m+1}x^{m+1} + C$
bx^m	$\frac{b}{m+1}x^{m+1} + C$

Merit and Excellence Questions

Merit/Excellence – Use calculus to solve each problem.

176. Vince enjoys sky diving. He has been told that from the moment he jumps from the plane to the point he releases his chute, his vertical acceleration is given by

$$a(t) = 8 - 0.2t \text{ m/s}^2$$

a) How long is it until he reaches his maximum velocity?

b) How fast is he travelling when he reaches his maximum velocity?

c) How far will he fall from the plane until he reaches this maximum velocity?

177. A curve has the equation

$$f(x) = ax^2 + bx - 3$$

and has a turning point at (2, 3)

a) Use calculus to find the values of 'a' and 'b'.

b) If the equation had been in the form

$$f(x) = ax^2 + bx + c$$

for what values of 'c' would the turning point be a maximum?

178. The gradient of a curve is given by

$$f'(x) = 6x^2 - 42x + 60$$

and it has a y-ordinate of 26 at its minimum turning point.

a) What are the x-ordinates of its turning points?

b) What is the y-ordinate of its maximum turning point?

Practice External Assessment Task

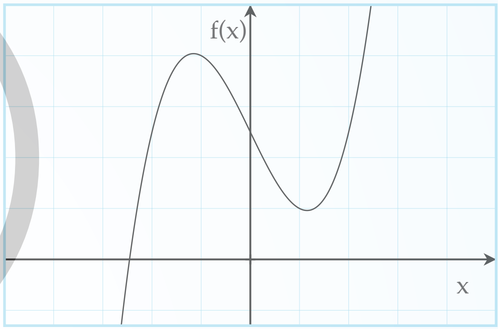
Calculus Methods 2.7

Make sure you show ALL relevant working for each question.

You are advised to spend 60 minutes answering this assessment.

QUESTION ONE

(a) (i) Find the gradient of $f(x) = x^3 - 4x + 5$ at $x = 2$.



(ii) Find the equation of the tangent to the curve $y = 3x - 3x^2 - x^3 + 4$ at the point $x = {}^-1$.

Answers

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1. At $x = {}^-2$, $m = {}^-2$

At $x = 2$, $m = 2$

2. At $x = {}^-4$, $m = 3$

At $x = {}^-1$, $m = {}^-2$

At $x = 1$, $m = 0$

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3. a) $m = 1$ at $(3, 2.5)$

b) $m = {}^-2$ at $({}^-3, 5.5)$

4. a) $m = 2$ at $x = {}^-3.5$ approx. and $x = 3.5$ approx.

b) $m = 0$ at $x = {}^-2$ and $x = 2$

5. a) Increasing

$x < {}^-4$ or $x > {}^-1$

b) Decreasing

${}^-4 < x < {}^-1$

6. At $x = {}^-4$, ${}^-1$ and 4

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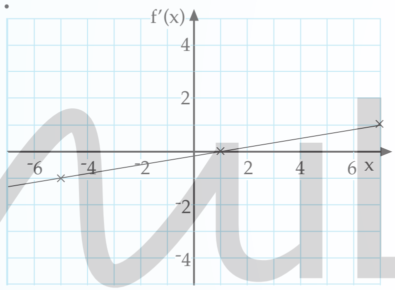

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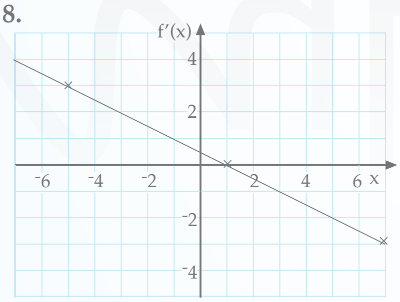

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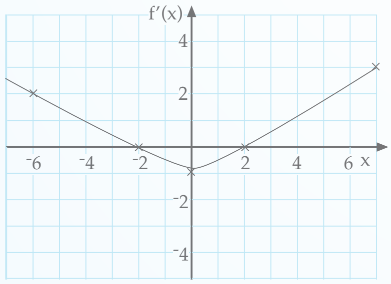

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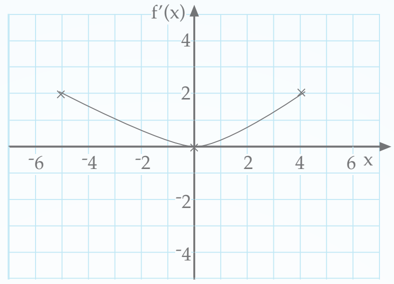

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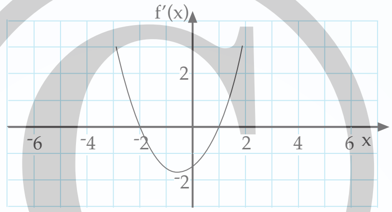

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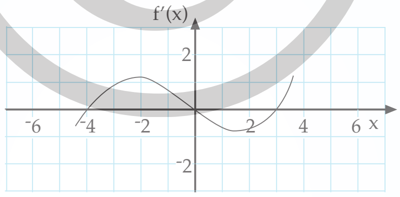

13.

Description	Looks like	Gradient function
Increasing		Always +ve
Decreasing		Always –ve
Maximum		Is zero. Gradient goes from +ve through 0 to –ve
Minimum		Is zero. Gradient goes from –ve through 0 to +ve

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14. Average rate of change = 4

15. Average rate of change = 3

16. Average rate of change = ${}^-2$

17. Average rate of change = ${}^-4$

18. Average rate of change = 4

19. Average rate of change = 3

20. Average rate of change = 2.4

21. Average rate of change = 2.01

22. Average rate of change = 2.001

23. Average rate of change = 2.0001

24. Avg. rate of change = 2.000 01

25. Avg. rate of change = 2.000 001

26. Expect the gradient to be 2

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27. $f'(x) = 2x + 2$

28. $f'(x) = 2x - 6$

29. $f'(x) = 5 - 2x$

30. $f'(x) = 2x + 3$

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31. $f'(x) = 3x^2$

32. $f'(x) = {}^-3$

33. $f'(x) = {}^-4x - 1$

34. $f'(x) = 6x^2 - 1$

35. $f'(x) = 20x$

36. $f'(x) = 8x - 5$

37. $f'(x) = 2x + 1$

38. $f'(x) = 2x - 7$

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39. $f'(x) = 15x^2$

40. $f'(x) = 9$

41. $f'(x) = 0$

42. $f'(x) = 2x + 3$

43. $f'(x) = 4x$

44. $f'(x) = 10x - 10x^4$

45. $f'(x) = 15x^2 + 4x$

46. $f'(x) = 10x + 10$

47. $f'(x) = 55x^{10} - 45x^4$

48. $f'(x) = x - 2$

49. $f'(x) = \frac{1}{2}x - \frac{1}{5}$

50. $f'(x) = 2x^2 - \frac{1}{4}$

51. $f'(x) = 2x^3 - 0.75x^2$

52. $f'(x) = 1.2x^5 + 0.9x^2 - 1.5$

53. $f'(x) = 1.5x^2 + 0.6x - 0.8$

54. $f'(x) = 6x^4 + 7x - 1.4$

55. $f'(x) = \frac{3x}{2} - \frac{1}{5} - \frac{x^2}{2}$

56. $f'(x) = \frac{10x^4}{3} - 3x^3 - \frac{6x^2}{5} + 8x - 2$

57. $f(x) = x^2 - 2x - 15$

$f'(x) = 2x - 2$

58. $f(x) = x^3 - 2x^2 + 5x - 10$

$f'(x) = 3x^2 - 4x + 5$

59. $f(x) = 3x^4 + 5x^3$

$f'(x) = 12x^3 + 15x^2$

60. $f(x) = x^4 - 2x^2 - 35$

$f'(x) = 4x^3 - 4x$